An Introduction to Higher Order Calculations in Perturbative QCD

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Overview of the Lectures

- Lecture I - Higher Order Calculations
  - What are they?
  - Why do we need them?
  - What are the ingredients and where do they come from?
  - Understanding and treating divergences
  - Examples from e^+e^- annihilation
- Lecture II - Examples of Higher Order Calculations
  - Parton Distribution Functions at higher order
  - Lepton Pair Production at higher order
  - Factorization scale dependence

- Lecture III - Hadronic Production of Jets, Hadrons, and Photons
  - Single inclusive cross sections
  - More complex observables and the need for Monte Carlo techniques
  - Overview of phase space slicing methods
- Lecture IV - Beyond Next-to-Leading-Order
  - When is NLO not enough?
  - Large logs and multiscale problems
  - Resummation techniques

Lecture IV - Outline

- When is an NLO calculation not really NLO?
- Use Lepton Pair Production as an example
  - $d\sigma/dQ^2$ versus $d\sigma/dQ^2\,dp_T$
  - NLO p_T distribution
- Two scale problems
- Large logs and the need for resummation
  - Leading double log approximation
  - k_T resummation – going beyond leading double logs
- Threshold resummation - an overview
- Applications
  - Hadron production
  - Dihadron production
  - Direct photon production
- Summary

When is an NLO Calculation not really NLO?

Recall our example of Lepton Pair Production from Lecture II where we calculated $d\sigma/dQ^2$

- LO $q\overline{q} \to l^+l^-$
- NLO $q\overline{q} \to l^+l^-$ (1-loop)
- NLO $q\overline{q} \to l^+l^-g$ and $qg \to l^+l^-q$ (tree graphs)
- Integrated over the additional variables for the radiated gluon or quark in the case of the $2 \to 3$ subprocesses
- Factorized the resulting collinear singularities and absorbed into the definition of the scale dependent PDFs
- PDFs with scale $M_f = Q$ interpreted as having the effects of radiated partons with p_Ts up to Q included

But what if we wanted to calculate $d\sigma/dQ^2\,dp_T$?

- The lowest order and the 1-loop virtual contributions are calculated with lowest order kinematics - at this stage the lepton pair has no p_T at the matrix element level
- The $\mathcal{O}(\alpha_S)$ tree graphs give the first non-zero lepton p_T at this stage of the calculation
- *But*, the graphs are convoluted with the bare PDFs

OK - what if we just change the bare PDFs to the scale dependent PDFs?

- Hmmm. We would then be including the effects of radiated parton p_Ts up to the value of the scale chosen for the PDFs
- But we are, at the same time, examining the p_T of the lepton pair which recoils against the radiated partons. Is there an inconsistency here?

No - not if we are content to calculate the p_T distribution at values of p_T which are of the order of Q

Lesson: the tree graphs which, after integrating over the recoiling parton p_T, contributed to the $\mathcal{O}(\alpha_S)$ correction for $d\sigma/dQ^2$ are now giving the LO contribution to the high Q tail of the p_T distribution

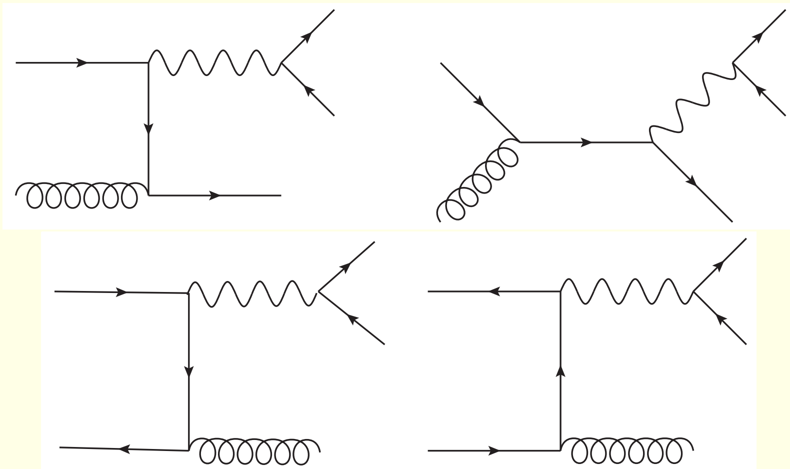
What if we wanted the NLO p_T distribution? We would have to do more!

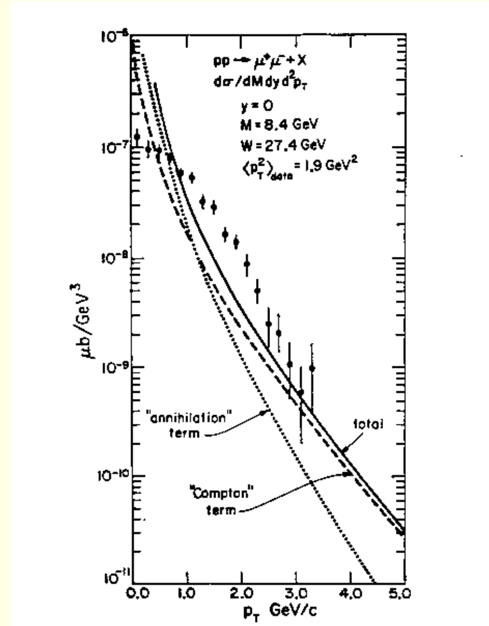
- Include the 1-loop corrections to the $\mathcal{O}(\alpha_s)$ tree graphs
- Also include the $\mathcal{O}(\alpha_s^2)$ tree graphs
- With these ingredients one could generate NLO predictions for the p_T distribution in the region where p_T is of the order of Q

But what if one was interested in the region where $p_T << Q$?

There are several problems

- As noted before, the scale dependent PDFs contain the contributions from integrating the radiated parton p_Ts up to $\mathcal{O}(Q)$ so there is a contradiction if we ask for the p_T of the lepton pair to be much less than Q
- There are now two scales in the problem - p_T and Q and if $p_T << Q$ one can encounter large logs of the ratio Q/p_T which should be resummed





The $\mathcal{O}(\alpha_S)$ subprocesses both give contributions which diverge as p_T^{-2} as p_T goes to zero

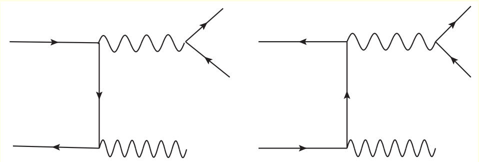
These divergent terms are factorized and included in the scale dependent PDFs

We want to do a better calculation in the low p_T region

- Have to figure out what to do with the low p_T radiated partons
- Have to figure out what the scale should be for the PDFs

Simple Example - $e^+e^- \rightarrow l^+l^- + \gamma$

- Discussion follows G. Parisi and R. Petronzio, Nucl. Phys. B154, 427 (1979)
- Avoids the complications due to the non-abelian nature of QCD and initial state PDFs, but illustrates the physics



- Cross section for fixed lepton pair mass Q diverges as k_T^{-2}, where k_T is the transverse momentum of the radiated photon
- For $k_T << Q$ we get

$$\frac{d\sigma}{dQ^2\, dp_T^2} = \frac{4\alpha^3}{3k_T^2} \frac{1}{SQ^2} \frac{S^2 + Q^4}{S - Q^2}$$

- Next, integrate over Q: $4m_\mu^2 < Q^2 < S - 2\sqrt{S}k_T$

- Letting $\sigma_0 = \frac{4}{3}\frac{\pi\alpha^2}{S}$ and performing the integration over Q^2 yields (keeping only the most divergent term)

$$\frac{d\sigma}{dk_T^2} = \sigma_0 \frac{\alpha}{\pi} \frac{\ln S/k_T^2}{k_T^2}$$

- Next, consider a partially integrated cross section defined as

$$\Sigma(k_T^2) = \frac{1}{\sigma_0} \int_0^{k_T^2} \frac{d\sigma}{dp_T^2}\, dp_T^2$$

- We know that there is a divergence at $p_T = 0$. But, we also know that the one loop corrections to our tree graphs will also contribute there and that if we were to integrate over all p_T we would get a finite result.
- To logarithmic accuracy we can write

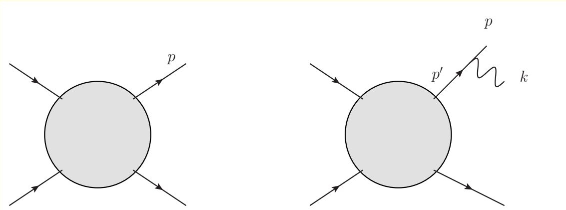
$$\frac{1}{\sigma_0} \int_0^{S} \frac{d\sigma}{dp_T^2}\, dp_T^2 = 1 + \mathcal{O}(\alpha) \times \text{constant} = \frac{1}{\sigma_0} \int_0^{k_T^2} \ldots + \frac{1}{\sigma_0} \int_{k_T^2}^{S} \ldots$$

- Therefore, we can write

$$\begin{aligned}\Sigma(k_T^2) &= 1 - \frac{1}{\sigma_0}\int_{k_T^2}^{S} \frac{d\sigma}{dp_T^2}\, dp_T^2 \\ &= 1 - \frac{\alpha}{\pi}\int_{k_T^2}^{S} \frac{dp_T^2}{p_T^2} \ln \frac{S}{p_T^2} \\ &= 1 - \frac{\alpha}{2\pi} \ln^2 \frac{S}{k_T^2}\end{aligned}$$

- Note that this is result is correct in the leading double log approximation. $\mathcal{O}(\alpha)$ terms which are constants or single logs are not included
- Now, what would happen if there were multiple photons emitted instead of just one?

Consider a process where there is a fermion in the final state and then compare it to one where there is a photon emitted from the fermion



$$\overline{u}(p) \to \overline{u}(p)\frac{\not{\epsilon}\not{p}'}{p'^2}$$

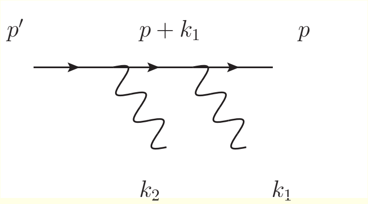
- Use $p' = p + k$ and use the fact that we are interested in soft photons - drop k everywhere except where it would lead to a divergence
- The factor associated with the photon emission is now

$$\overline{u}(p)\frac{\not{\epsilon}\not{p}}{2p \cdot k}$$

- Using the Dirac equation, this may be simplified to

$$\overline{u}(p)\,\frac{p\cdot\epsilon}{p\cdot k}$$

- Now, what about two soft photons?



- Repeat the above analysis and symmetrize the result by interchanging the two photons and dividing by two
- The result is a factor

$$\frac{1}{2}\,\frac{p\cdot\epsilon_1}{p\cdot k_1}\,\frac{p\cdot\epsilon_2}{p\cdot k_2}$$

- Similarly, for n terms one gets

$$\frac{1}{n!}\frac{p\cdot\epsilon_1}{p\cdot k_1}\cdots\frac{p\cdot\epsilon_n}{p\cdot k_n}$$

- Soft photon emission factorizes!
- For n emissions we get a contribution to the cross section

$$\frac{1}{\sigma_0}d\sigma = \frac{\alpha^n}{n!}dk_{T1}^2\ldots dk_{Tn}^2\nu(k_{T1})\ldots\nu(k_{T2})$$

- where $\nu(k_T) = \frac{\ln S/k_T^2}{k_T^2}$ is the result for a single photon
- We want to calculate the contribution of the n photon term to the integrated p_T distribution given by $\Sigma(k_T^2)$
- As a first attempt, ignore the correlations between the transverse momenta of the emitted photons - treat them all as being independent

- Then the n^{th} term is just

$$\begin{aligned}
\Sigma^{(n)}(k_T^2) &= \frac{1}{n!}\left[\int_0^{k_T^2} dp_T^2\,\frac{\alpha}{\pi}\frac{\ln S/p_T^2}{p_T^2}\right]^n \\
&= \frac{1}{n!}\left[1-\frac{\alpha}{2\pi}\frac{\ln^2 S/k_T^2}{k_T^2}\right]^n \\
&= \frac{(-1)^n}{n!}\left(\frac{\alpha}{2\pi}\ln^2 S/k_T^2\right)^n+\cdots
\end{aligned}$$

- Summing over all n yields

$$\Sigma(k_T^2) = \exp(-\frac{\alpha}{2\pi}\ln^2 S/k_T^2)$$

Next, we can recover the differential cross section by taking a derivative of Σ

$$\begin{aligned} \frac{1}{\sigma_0} \frac{d\sigma}{dk_T^2} &= \frac{d}{dk_T^2} \Sigma(k_T^2) \\ &= \frac{\alpha}{\pi} \frac{\ln S/k_T^2}{k_T^2} exp\left(-\frac{\alpha}{2\pi} \ln^2 S/k_T^2\right) \end{aligned}$$

- Notice that as $k_T \to 0$ the differential cross section now vanishes, rather than diverges
- Summing the leading double logs has tamed the divergence, but at the price of a vanishing cross section. The suppression is too strong, as we will see shortly

  (Exercise: fill in the steps for the derivation of this result)

Interpretation

- Σ is referred to as a Sudakov form factor and it can be interpreted as given the probability for emitting no photons with transverse momenta greater than k_T
- We enforced the independent emission hypothesis and neglected conservation of transverse momentum. The only way to get a lepton pair with zero transverse momentum, was to suppress *all* photon emission.
- The probability of emitting no photons in a collision which creates a massive lepton pair is zero
- The lowest order divergence is actually just the first term in an expansion of the exponential which vanishes at zero transverse momentum

How can we restore transverse momentum conservation?

Insert a δ function which enforces conservation of transverse momentum for the emission of n photons

$$\frac{1}{\sigma_0}\frac{d\sigma^N}{d^2p_T} = \frac{1}{n!}\left(\frac{\alpha}{\pi}\right)^n \int d^2k_{T_1}\dots d^2k_{T_n}\nu(k_{T_1})\dots\nu(k_{T_n})\delta^2(\vec{p}_T - \vec{k}_{T_1} - \dots - k_{T_n})$$

Next, use the Dirac representation of the δ function

$$\delta^2(\vec{p}_T - \vec{k}_{T_1} - \dots - k_{T_n}) = \frac{1}{(2\pi)^2}\int d^2b e^{-i\vec{b}\cdot(\vec{p}_T - \vec{k}_{T_1} - \dots - \vec{k}_{T_n})}$$

- Notice how the integrand still factorizes, even with the δ function included
- Define the Fourier transform of ν by

$$\tilde{\nu}(b) = \frac{1}{\pi}\int d^2k_T e^{i\vec{b}\cdot\vec{k}_T}\nu(k_T)$$

- The n photon emission contribution now looks like

$$\frac{1}{\sigma_0}\frac{d\sigma}{d^2p_T} = \frac{\alpha^n}{4\pi^2 n!}\int d^2 b e^{-i\vec{b}\cdot\vec{p}_T}\,[\tilde{\nu}(b)]^n$$

- We see that the the exponentiation can now take place in impact parameter space

$$\frac{1}{\sigma_0}\frac{d\sigma^n}{d^2p_T} = \frac{1}{4\pi^2}\int d^2 b e^{-i\vec{b}\cdot\vec{p}_T}\,\tilde{\sigma}(b)$$

where $\tilde{\sigma}(b) = exp[\alpha\tilde{\nu}(b)]$

(Exercise: fill in the steps to derive this result)

Interpretation

- In the first case, the Sudakov form factor entered because we demanded that we approach zero transverse momentum of the lepton pair by limiting the transverse momenta of *all* the emitted photons individually
- By inserting the transverse momentum conserving delta function and exponentiating in impact parameter space, we allowed for the possibility of two or more photons balancing in transverse momentum and giving a zero result
- Formally, these terms are subleading, but the leading terms vanish and so the subleading terms become dominant

Extension to QCD

- This concept was extended to QCD by Collins, Soper, and Sterman (Nucl.Phys.B250,199(1985))

- Need to take into account the transverse momentum of the incoming quarks
  - Normally integrated over, leading to the scale dependence of the PDFs
  - Factorization scale usually chosen to be on the order of the single hard scale
  - Now, the lepton pair p_T will reflect the p_Ts of the incoming quarks
  - PDF scale is chosen to be of the order of $1/b$ where b is the impact parameter seen above
  - b and p_T are conjugate variables - large $p_T \leftrightarrow$ small b
  - A scale of $1/b$ is large for large p_T and small for small p_T
- Classic application is to the lepton pair, W, or Z p_T distributions

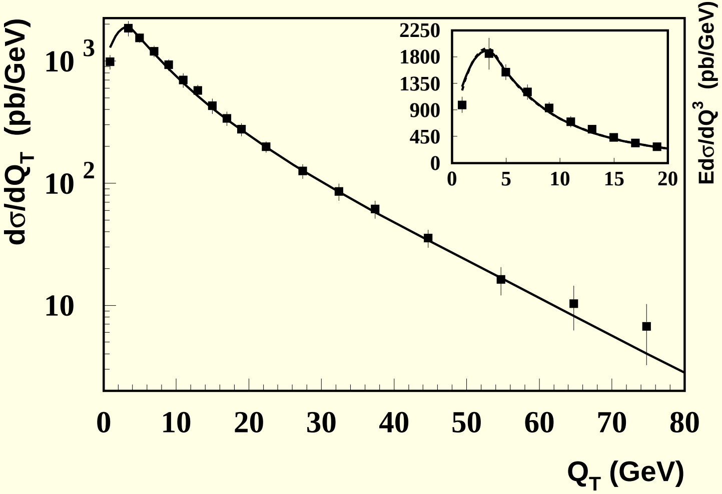
CSS Resummed Result

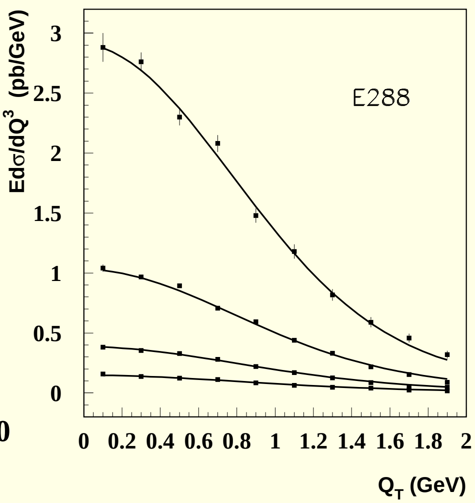
The resummed CSS result takes a relatively simple form with an exponentiation in impact parameter space and a convolution with PDFs evaluated at a scale $1/b$

$$\begin{aligned}\frac{d\sigma}{dQ^2\,dy\,dk_T^2} &= \frac{4\pi^2\alpha^2}{9Q^2S}(2\pi)^{-2}\int d^2b e^{i\vec{k}_T\cdot\vec{b}}\sum_j e_j^2 \\ &\sum_a\int_{x_a}^1\frac{d\xi_a}{\xi_a}G_{a/A}(\xi_a,1/b)\sum_b\int_{x_b}^1\frac{d\xi_b}{\xi_b}G_{b/B}(\xi_b,1/b) \\ &e^{-S(Q^2,b)}C_{ja}(\frac{x_a}{\xi_a},g(1/b))C_{\bar{j}b}(\frac{x_b}{\xi_b},g(1/b)) \\ &+ \frac{4\pi^2\alpha^2}{9Q^2S}Y(k_T,Q,x_a,x_b)\end{aligned}$$

with $S(Q^2,b) = exp\left[-\int_{1/b^2}^{Q^2}\frac{d\overline{\mu}^2}{\overline{\mu}^2}\left[\ln\left(\frac{Q^2}{\overline{\mu}^2}\right)A(g(\overline{\mu}))+B(g(\overline{\mu}))\right]\right]$

- The Y piece is the residual NLO non-log contribution
- One can see the resemblance to my earlier example modified by the inclusion of the scale dependent PDFs
- In the expression for S, the A term sums the leading logarithms while the B term sums the next-to-leading logs
- Here are some typical resummed results (from J. Qiu and X. Zhang, Phys. Rev. D63:114011,2001) compared to data (D0 and Fermilab E-288)





- Note that by exponentiating in impact parameter space $\frac{d\sigma}{dk_T^2}$ has a non-zero intercept at $k_T = 0$
- The D0 data are shown as $\frac{d\sigma}{dk_T}$ which has a kinematic zero at $k_T = 0$
- For both plots, however, the tree level calculation would diverge as $k_T \to 0$, whereas the b-space exponentiation describes the data nicely

Other Resummation Examples

Logarithms of variables other than k_T can also occur - it depends on the type of distribution one is calculating. The logs come from the same basic vertices in the Feynman diagrams - they just appear in different ways and require different types of treatments. Another example is provided by the threshold logs we encountered previously in Lecture II

Threshold Resummation – Basic Physics

- For inclusive calculations, singularities from soft real gluon emission cancel against infrared singularities from virtual gluon emission

- Limitations on real gluon emission imposed by phase space constraints can upset this cancellation

- Singular terms still cancel, but there can be large logarithmic remainders

- Classic example is thrust distribution in $e^+e^- \to jets$

Consider the thrust distribution for small values of thrust, retaining the most divergent term

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_s}{2\pi} C_F \frac{4}{1-T} \ln \frac{1}{1-T}$$

- We recognize that as $T \to 1$ the phase space for gluon emission is being restricted
- We can use the same technique we used for the k_T example - split the integral over T into two pieces
- Let $f(T) = \int_T^1 dT\, \frac{1}{\sigma}\frac{d\sigma}{dT}$
- Then we can write

$$\int_{T_{min}}^{1} dT \frac{1}{\sigma}\frac{d\sigma}{dT} = \int_{T_{min}}^{T} dT\, \frac{1}{\sigma}\frac{d\sigma}{dT} + f(T) = 1 + \mathcal{O}(\alpha_S) \times \text{constant}$$

- Hence, $f(T)$ is given by

$$\begin{aligned} F(T) &= 1 - \int_{T_{min}}^{T} dT\,(-4)\frac{\alpha_s}{2\pi} C_F \frac{\ln(1-T)}{1-T} \\ &= 1 - \frac{\alpha_S}{\pi} C_F \ln^2(1-T) \end{aligned}$$

- In the leading double log approximation this can be exponentiated to give

$$f(T) = e^{-\frac{\alpha_s}{\pi} C_F \ln^2(1-T)}$$

- Taking a derivative yields

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = -2\frac{\alpha_S}{\pi} C_F \frac{\ln(1-T)}{1-T} e^{-\frac{\alpha_s}{\pi} C_F \ln^2(1-T)}$$

(Exercise: fill in the steps for this derivation)

- This gives a turnover as $T \to 1$ which cures the divergence seen back in Lecture I
- However, the turnover occurs at $\alpha_s \ln(1-T) = \frac{\pi}{2C_F} \approx 1.2$, violating the assumption that $\alpha_s \ln(1-T) << 1$ (Exercise: show this)
- Must go further and include additional log terms, not just the double logs
- However, this simple example serves as an introduction to the idea of threshold resummation

More General Processes

- The general structure of more complex cross sections will be a generalization of what we have already seen
  - Collinear singularities associated with the incoming and outgoing partons - these are collected into *jet functions*
  - Soft singularities associated with the emission of low-energy gluons - these are collected into an overall *soft function*
  - A hard scattering remainder that is finite

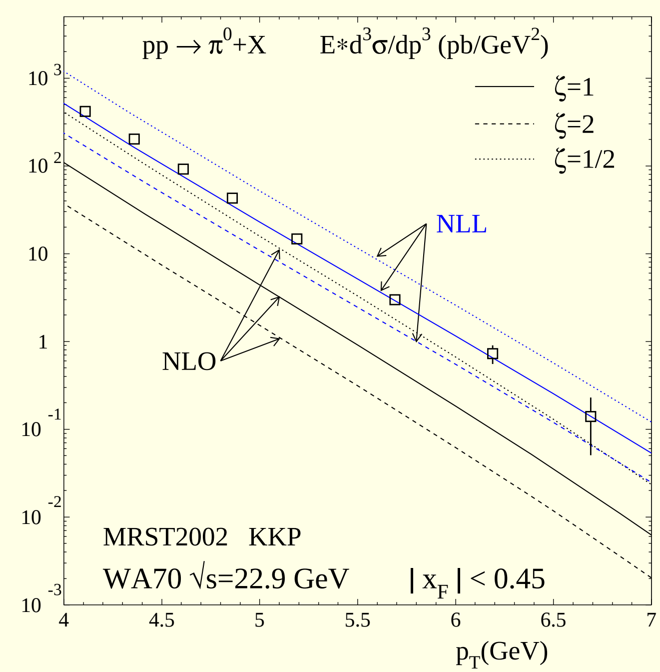
- The jet functions that describe the collinear singular region will contain the "plus" distributions that we have already seen that describe the real/virtual cancellations
- For a given observable one must show that
  - The squared amplitude factorizes in the manner described above
  - Phase space factorizes
  - This will generally require some type of transform like the Fourier transform we used in the k_T case
  - For threshold resummation this is usually a Mellin transform involving moments of the cross section with respect to a large scaled energy variable, *e.g.*, $\tau = Q^2/S$ for lepton pair production
  - The physical cross section then obtained by taking the inverse transformation after the exponentiation

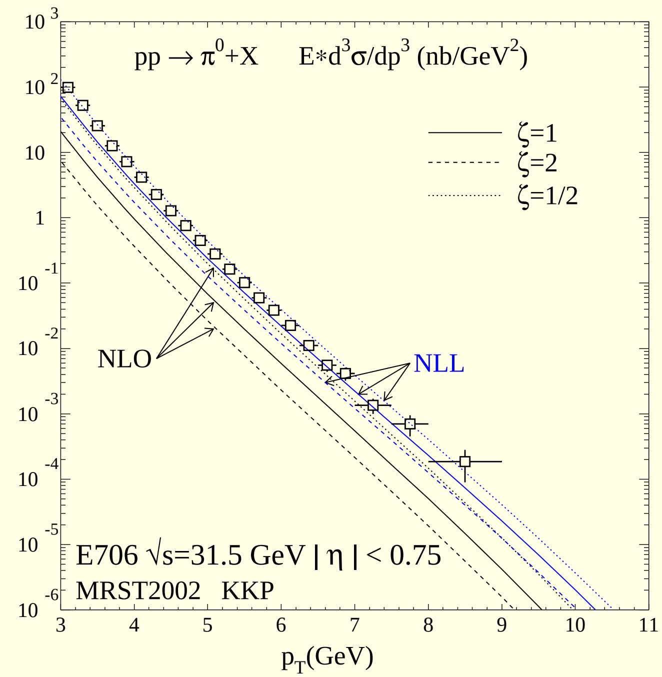
Example – High-p_T/high-mass particle production

- Phase space for gluon emission is limited near threshold in the parton-parton scattering subprocess
- Steeply falling PDFs constrain real gluon emission when high-mass or high-p_T is required
- Expect threshold resummation to be important as $x_T \to 1$
- Define $v = 1 + t/s$ and $w = -u/(s+t)$.
- Threshold occurs at $w = 1 \;\; (s + t + u = 0)$
- $\frac{d\sigma}{dv\,dw}$ contains terms like $\alpha_s^m \left(\frac{\ln^n(1-w)}{1-w}\right)_+$
- Can sum leading logs $(n = 2m - 1)$, next-to-leading-logs $(n = 2m - 2)$, etc.
- "Threshold resummation is resummation of the 'plus' distributions"

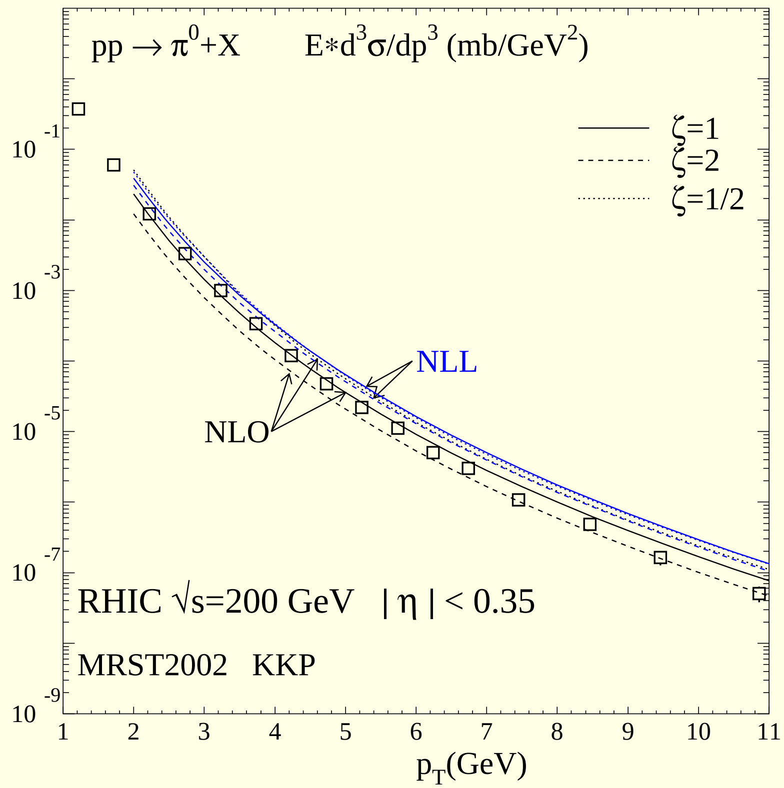
High-p_T π^0 cross section

- Work by de Florian and Vogelsang (hep-ph/0501258) applies threshold resummation to π^0 production
- For fixed target experiments the center of mass energy is in the 20-40 GeV range while p_T is typically 3-12 GeV $\Rightarrow x_T$ can be large
- The fragmentation fraction z (see Lecture I) will also be large when one requires a high-p_T hadron - the hadron will take most of the energy of the fragmenting parton (jet) since taking a smaller fraction wastes energy and the parton PDFs fall off rapidly in x (nature doesn't want to waste the available partonic center of mass energy)
- Large values of z relevant for fixed target energies leads to large threshold resummation corrections $(\ln^n(1-z)/(1-z))$
- Enhancement is strongly energy dependent since the relevant values of z decrease as one goes to higher energies at fixed p_T (more energy is available at fixed p_T and the relevant values of z decrease)





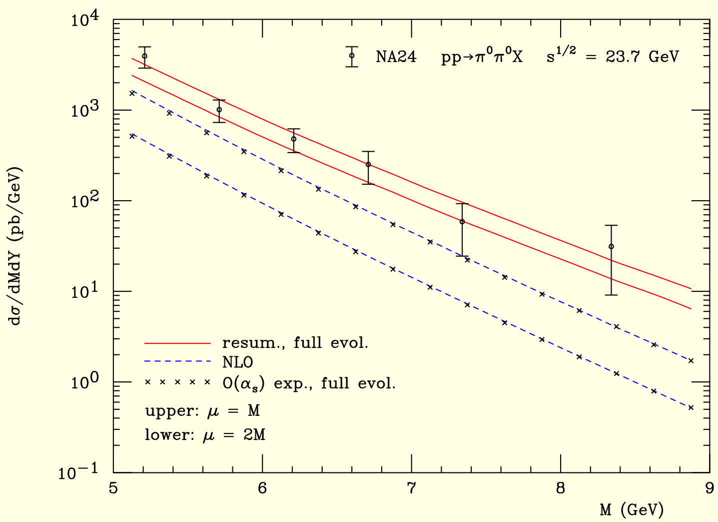
- Blue curves include the resummation corrections properly matched to an existing NLO calculation in order to avoid double counting
- Note the reduced scale dependence of the resummed results

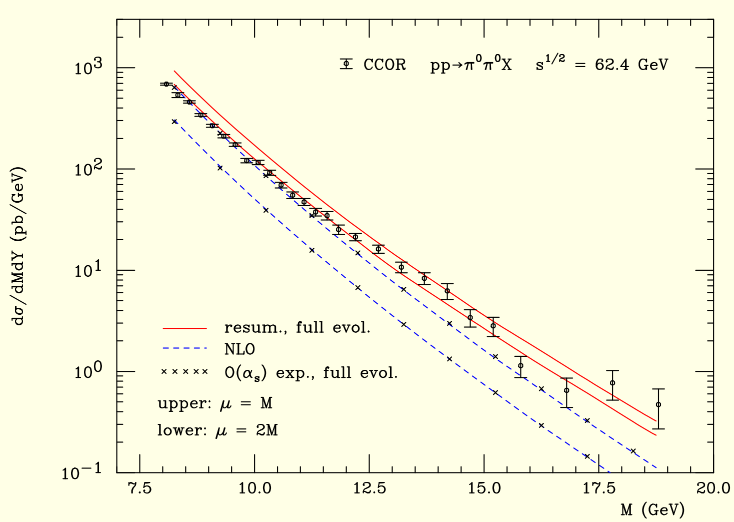


- Note reduced enhancement at RHIC energy compared to the previous fixed target results

Another Example – Dihadron Production

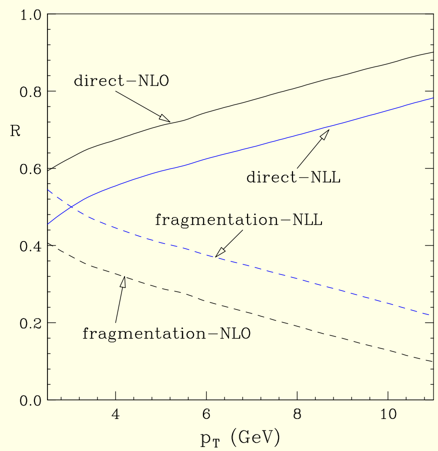
- Two PDFs and two FFs means there are four jet functions
- In the region of large M^2/S where M is the dihadron mass we expect large threshold logs
- Recent work by Almeidea, Sterman, and Vogelsang (Phys.Rev.D80:074016,2009) confirms that the effects are large for fixed target experiments and that they decrease as S is increased at fixed M
- Results also show decreased scale dependence

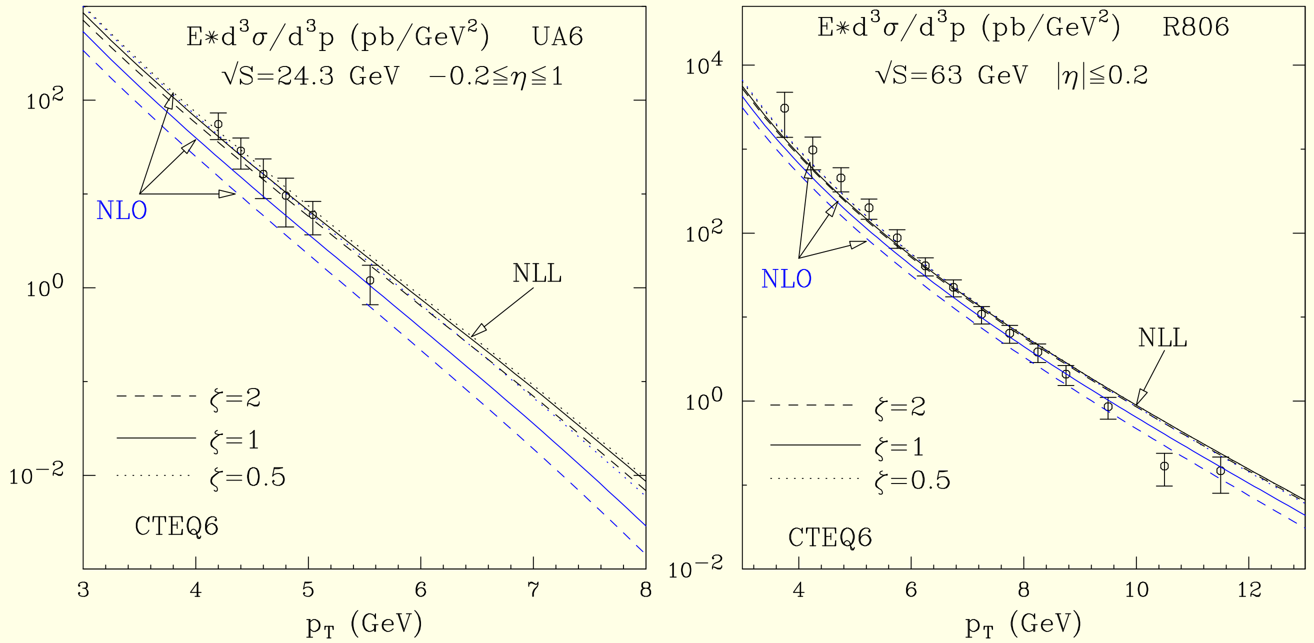




Direct Photon Production

- Same formalism applied to direct photon production has some interesting features (Vogelsang and de Florian, Phys.Rev. D72 (2005) 014014)
  - Direct subprocesses $q\overline{q} \to \gamma g$ and $qg \to \gamma q$ are not enhanced significantly
  - Fragmentation subprocesses where the photon is radiated from an outgoing quark, for example, are significantly enhanced as is the case for hadron fragmentation processes
  - End result is a large increase for predictions at fixed target energies with a rapid fall off as the energy is increased at fixed p_T





- Also see the expected decrease in the scale dependence
- Threshold resummation improves the agreement with the fixed target experiments without adversely affecting the description of the high energy collider data

Summary

Here are some key points to remember

- NLO calculations are not always adequate for every observable
- NLO calculations are appropriate for processes where there is one large scale
- In some regions of phase space the NLO corrections can actually become LO if the lowest order contribution is suppressed
- Large logs can be generated when the phase space for additional gluon radiation is restricted - look for two or more relevant scales
- Examples include low-p_T lepton pair and vector boson production
- Threshold logs can be resummed and are especially relevant for understanding the energy dependence of fixed target hadron or photon production relative to the high energy collider results
- There is more to perturbation theory than just the next order contribution!