Perturbative QCD: the Basics II

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Outline

- I. General Introduction: Brief History and Basics of Basics
- II. Singularities and Infrared Safety
- IV. Factorization, Dedicated Example
- V. Example where factorization breaks

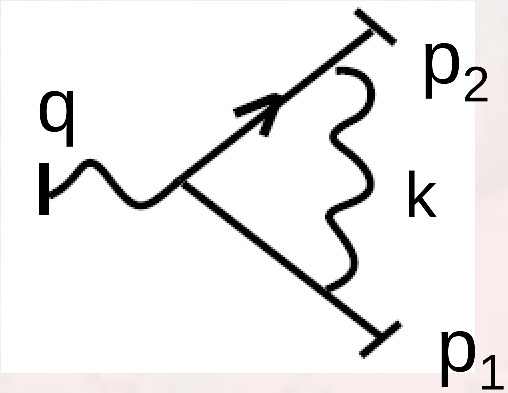
Infrared singularity and Infrared safety

- Important job in the factorization is to identify these infrared divergences
  - Kinematic region
- We have to show the cancellations for infrared safe quantities
- The residue infrared divergences can be handled carefully in the factorization

- In particular, a next-to-leading order calculation will be very helpful to demonstrate the factorization and infrared safety in perturbative QCD
- It is also important that this can be generalized to all orders

Infrared structure and Landau equation

- Take the vertex loop as an example



$$I_\Delta = g^3 \mu^{3\epsilon} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + i\epsilon)((p_1 - k)^2 + i\epsilon)\ ((p_2 + k)^2 + i\epsilon)}$$

- Introduce Feynman parameters

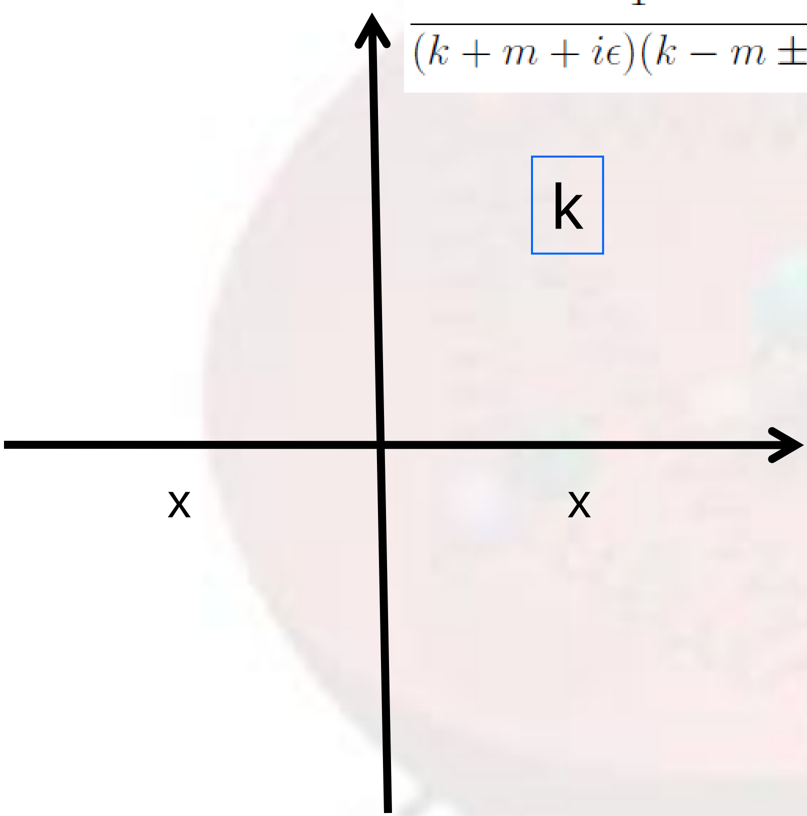
$$I_\Delta = 2\int \frac{d^n k}{(2\pi)^n} \int_0^1 \frac{d\alpha_1 d\alpha_2 d\alpha_3\, \delta(1-\sum_{i=1}^3 \alpha_i)}{D^3}$$

$$D = \alpha_1 k^2 + \alpha_2 (p_1 - k)^2 + \alpha_3 (p_2 + k)^2 + i\epsilon$$

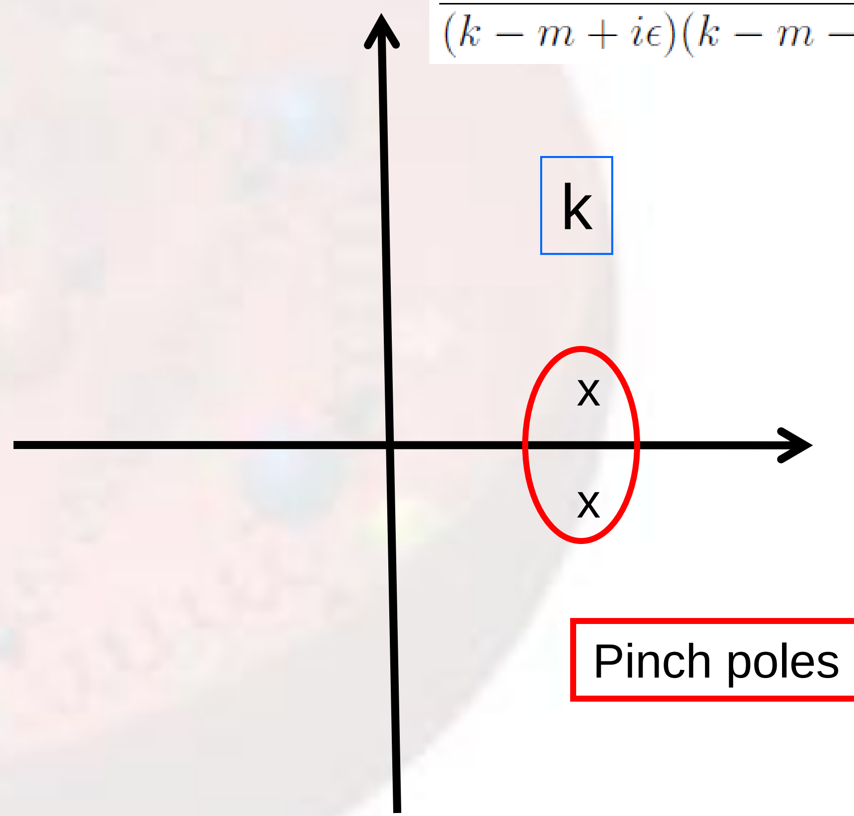
  - Singularities at D=0, there are two poles for each momentum integral
  - If the two poles are separated in the real axis, we can deform the contour, no singularity
  - Pinch pole : where the two poles are opposite half planes, we can not deform the contour → singularity

Examples

$$\frac{1}{(k+m+i\epsilon)(k-m\pm i\epsilon)}$$



$$\frac{1}{(k-m+i\epsilon)(k-m-i\epsilon)}$$



Landau equations

- Landau equations determine the location of the pinch singular surfaces

$$\frac{\partial}{\partial k^\mu} D(\alpha_i, k^\mu, p_a) = 0$$

$$\begin{aligned} \text{either } \ell_i^2 = m_i^2, \text{ or } \alpha_i &= 0, \\ \text{and } \sum_{i \,\epsilon\, \text{loop } s} \alpha_i \ell_i \epsilon_{is} &= 0, \end{aligned}$$

  - Vertex diagram

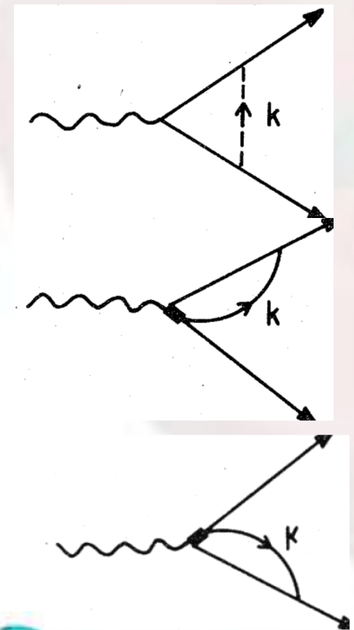
$$\alpha_1 k^\mu - \alpha_2 (p_1 - k)^\mu + \alpha_3 (p_2 + k)^\mu = 0 .$$

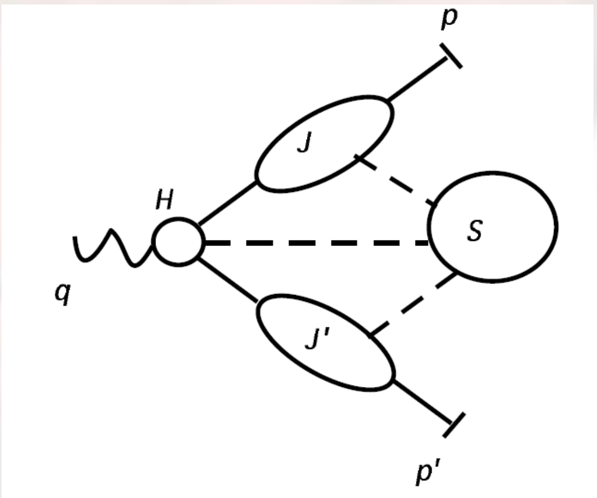
  - Three solutions:
    - Soft $\quad k^\mu = 0\,, \quad (\alpha_2/\alpha_1) = (\alpha_3/\alpha_1) = 0$
    - Collinear

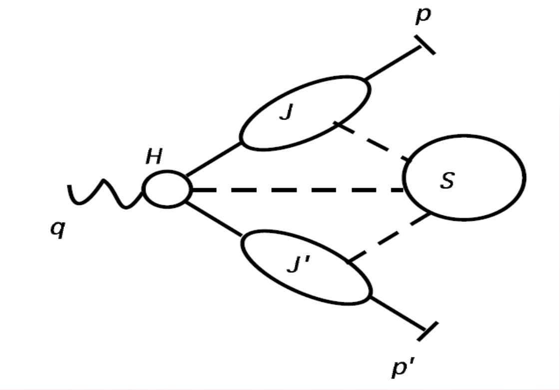
$$\begin{aligned} k &= \zeta p_1\,, \quad \alpha_3 = 0\,, \quad \alpha_1 \zeta = \alpha_2 (1 - \zeta) \\ k &= -\zeta' p_2\,, \quad \alpha_2 = 0\,, \quad \alpha_1 \zeta' = \alpha_3 (1 - \zeta') \end{aligned}$$

Reduced diagram and the Physical picture

- Coleman and Norton: The pinch singular surfaces correspond to the real space-time scattering picture



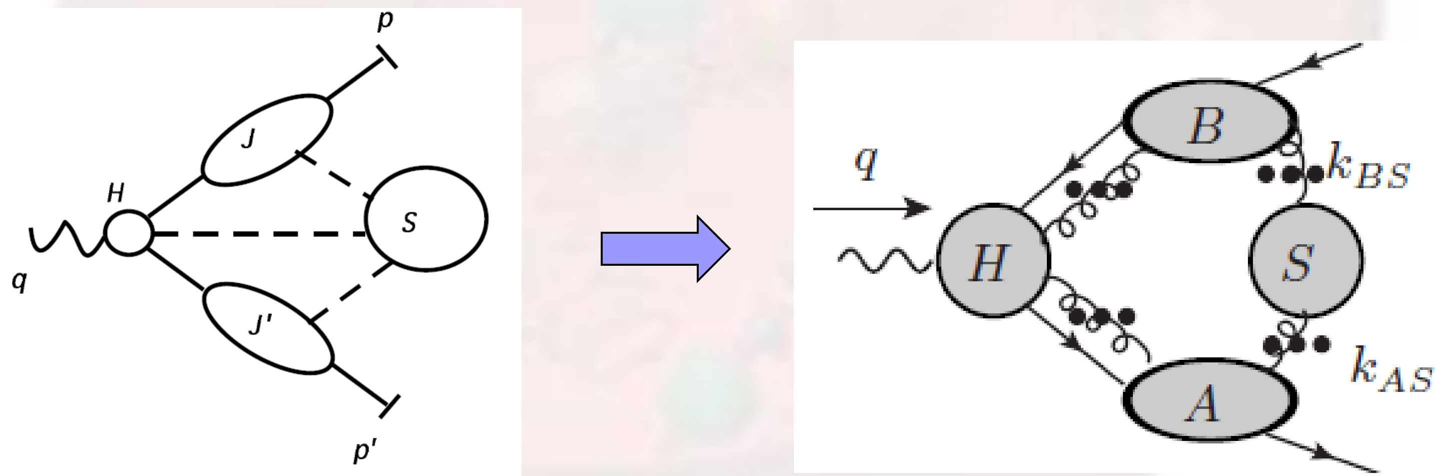




- Virtual photon decays into two jets: collinear particles along p and p' directions
- There are soft momenta connections between hard part and the two jets

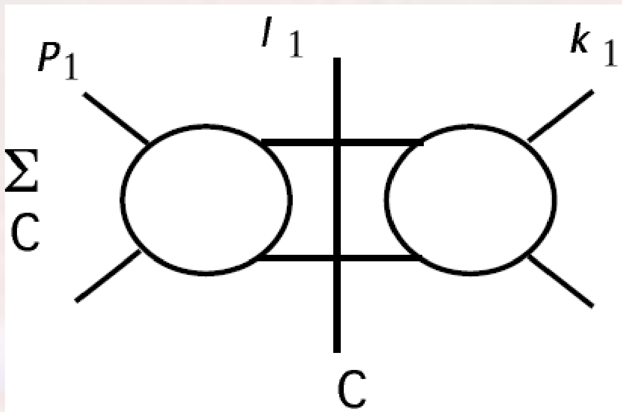
Power counting and the Leading region

- Leading contributions are determined from the power counting



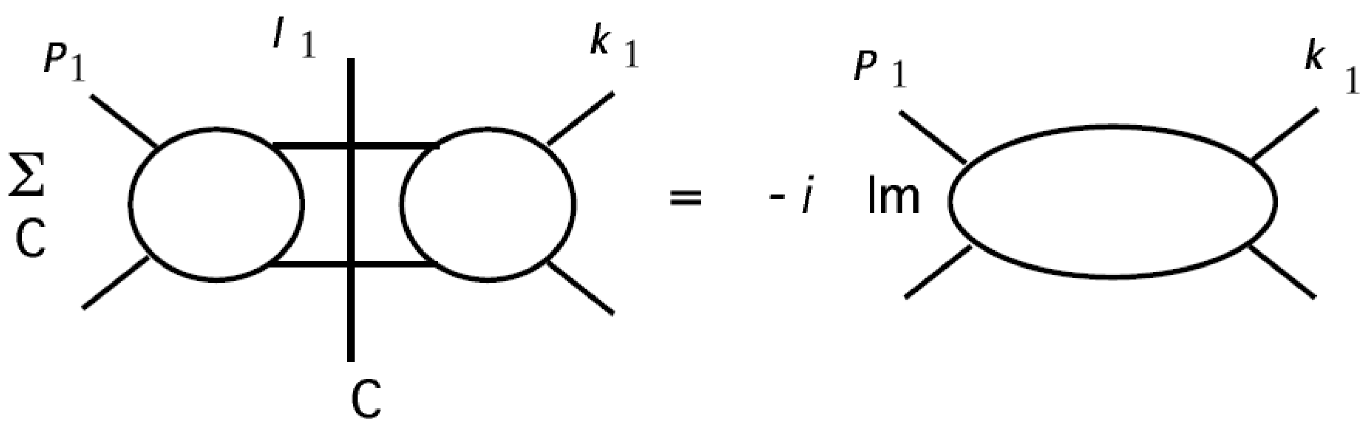
Unitarity and Infrared Safe

- Introduce the Cut diagrams



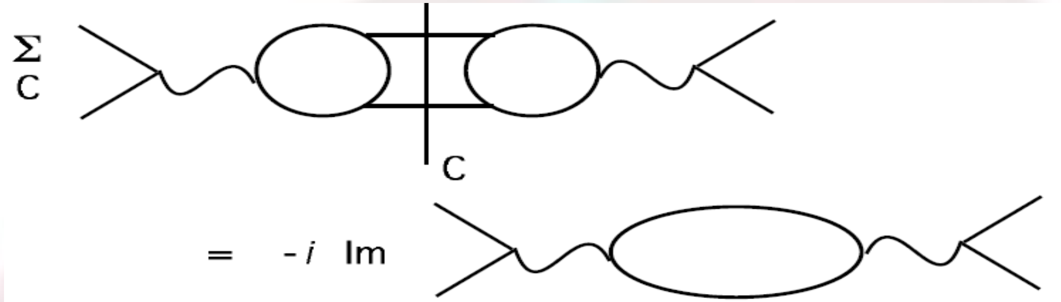
  - Amplitude of $p_1, \ldots, p_n$ scatter into $l_1, \ldots, l_n$, times the complex conjugate of $l_1, \ldots, l_n$ scatter into $k_1, \ldots, k_n$

- Generalized unitarity



  - Sum over all cuts with fixed external momenta $p_1,\ldots, p_n$ and $k_1,\ldots, k_n$ equals to the imaginary part of un-cut diagram

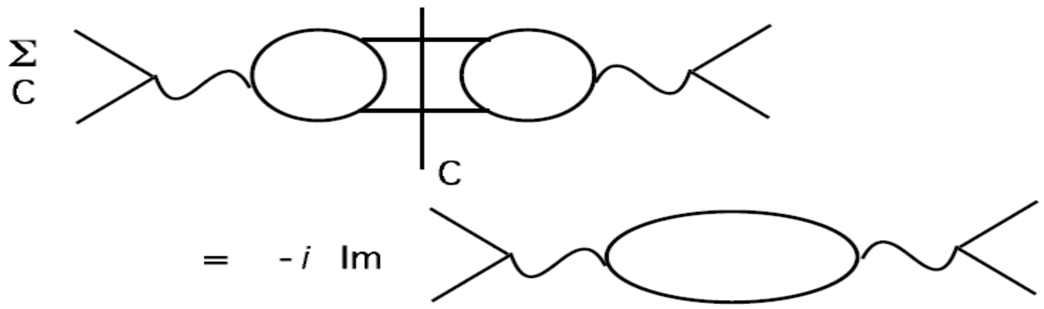
Apply to e+e- annihilation



- By optical theorem, the total cross section equals to the imaginary part of the forward scattering amplitude

$$\sigma^{(\mathrm{tot})}_{e^+e^-}(q^2) = \frac{e^2}{q^2} \mathrm{Im}\, \pi(q^2)$$

$$\pi(q^2)(q_\mu q_\nu - q^2 g_{\mu\nu}) = i \int d^4x \, e^{iqx} < 0|\, T\; J_\mu(x) J_\nu(0)\; |0 >$$

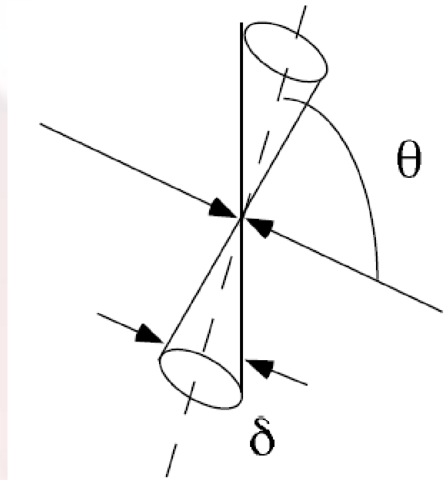


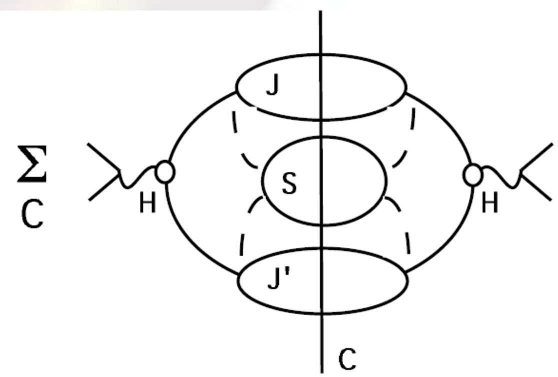
- There are no physical processes in which virtual photon decays into a set of on-shell particles, propagate freely and annihilate into the same mass photon
- This eliminates any pinch surfaces with finite momentum particles, and the infrared divergences in the total cross section

KLN Theorem

Weighted cross sections in e+e-

- Dijet production
  - Fraction of energy emitting into back-to-back two cones
  - Using the generalized unitarity, there is no pinch surface at all
  - Again infrared safe
  - Can be generalized to three-, four-jet, …

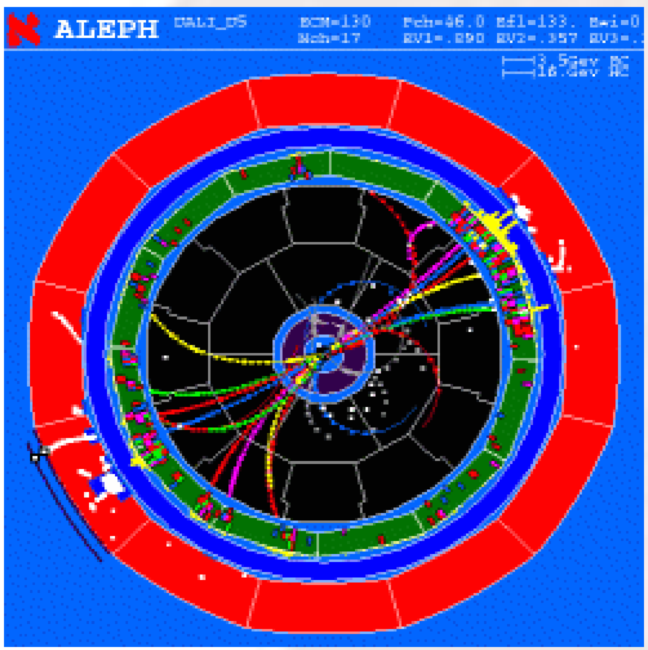




Example from the LEP

- Two-jet event

Three-jet event



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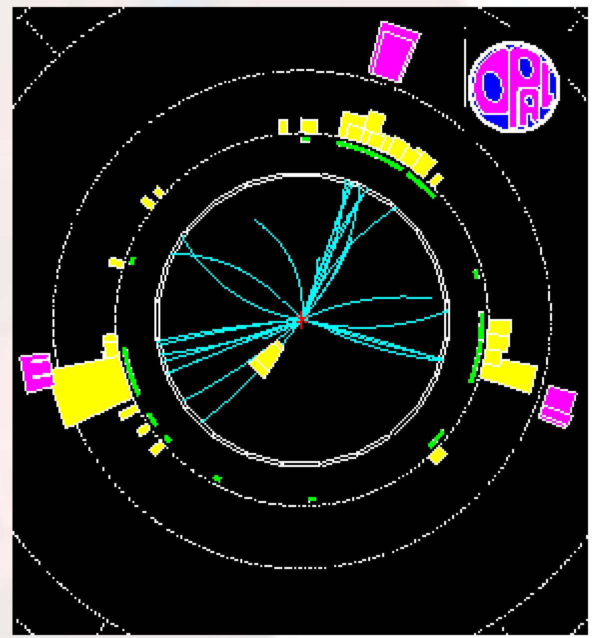

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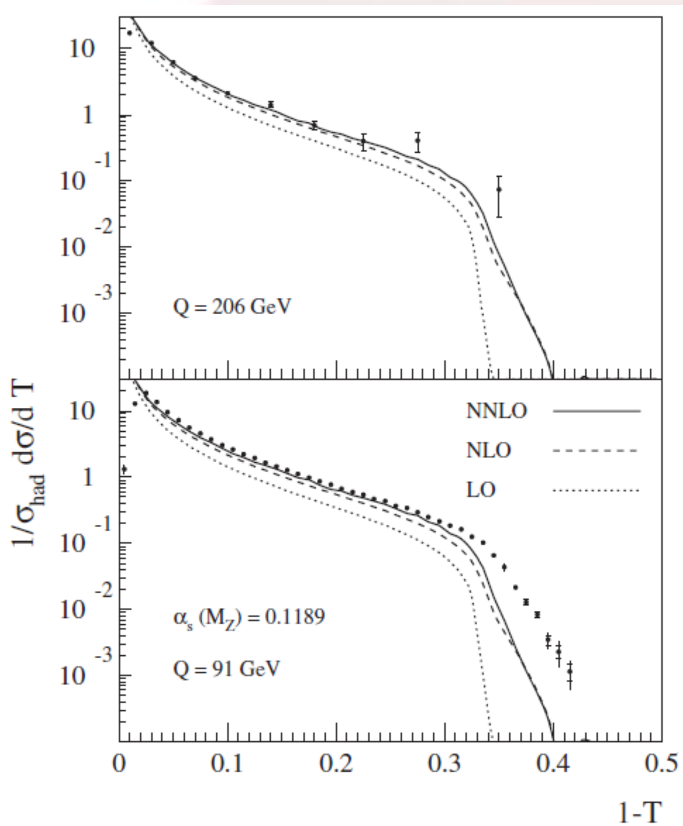
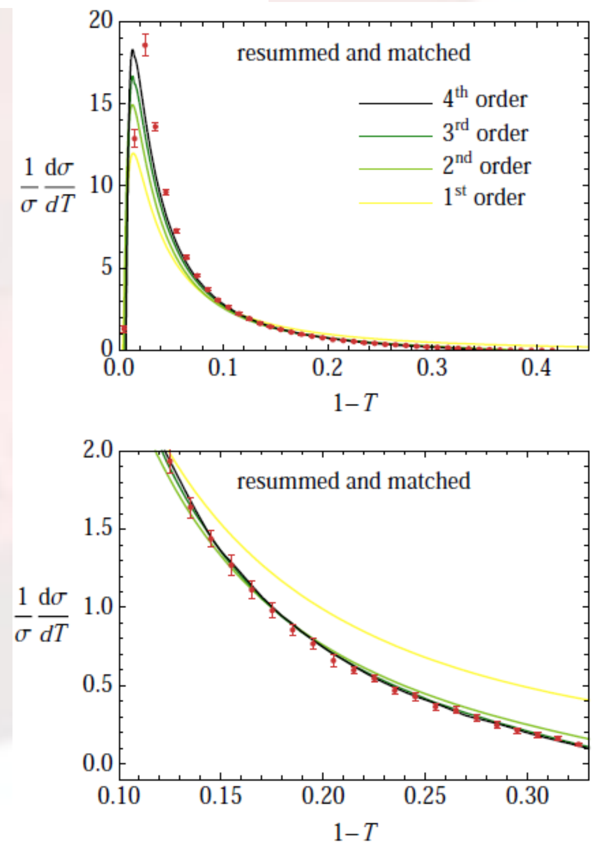
Event shape

$$\sigma_S = \sum_n \int d\tau_n \, \frac{d\sigma}{d\tau_n} \, \mathcal{S}_n(p_1 \ldots p_n)$$

- Thrust distribution

$$T = \frac{1}{\sum_i |\, \vec{p}_i \,|} \, \max_{\hat{n}} \sum_{i=1}^{n} |\, \vec{p}_i \cdot \hat{n} \,|$$



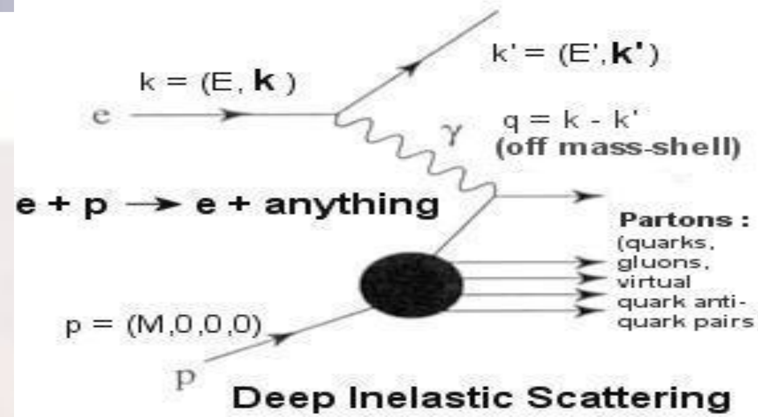


Long distance physics (factorization)

- Not every quantities calculated in perturbative QCD are infrared safe
  - Hadrons in the initial/final states, e.g.
- Factorization guarantee that we can safely separate the long distance physics from short one
- There are counter examples where the factorization does not work

Back to DIS

- Kinematics



$\nu = \dfrac{q \cdot P}{M} = E - E'$ is the lepton's energy loss in the nucleon rest frame (in earlier literature sometimes $\nu = q \cdot P$). Here, E and E' are the initial and final lepton energies in the nucleon rest frame.

$Q^2 = -q^2 = 2(EE' - \vec{k} \cdot \vec{k}\,') - m_\ell^2 - m_{\ell'}^2$ where $m_\ell (m_{\ell'})$ is the initial (final) lepton mass. If $EE' \sin^2(\theta/2) \gg m_\ell^2, m_{\ell'}^2$, then

$\approx 4EE' \sin^2(\theta/2)$, where θ is the lepton's scattering angle with respect to the lepton beam direction.

$x = \dfrac{Q^2}{2M\nu}$ where, in the parton model, x is the fraction of the nucleon's momentum carried by the struck quark.

$y = \dfrac{q \cdot P}{k \cdot P} = \dfrac{\nu}{E}$ is the fraction of the lepton's energy lost in the nucleon rest frame.

$W^2 = (P + q)^2 = M^2 + 2M\nu - Q^2$ is the mass squared of the system X recoiling against the scattered lepton.

$s = (k + P)^2 = \dfrac{Q^2}{xy} + M^2 + m_\ell^2$ is the center-of-mass energy squared of the lepton-nucleon system.

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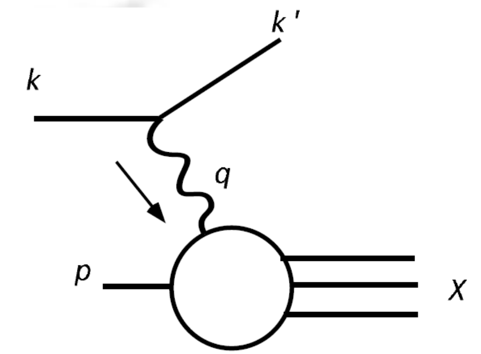
Structure functions (cross section)



- EM factorization (photon exchange)

$$d\sigma = \frac{d^3k'}{2s|\vec{k}'|}\,\frac{1}{(q^2)^2}\,L^{\mu\nu}(k,q)W_{\mu\nu}(p,q)$$

$$L^{\mu\nu} \equiv \frac{e^2}{8\pi^2} tr\left[\not{k}\gamma^\mu \not{k}'\gamma^\nu\right]$$

- Hadronic tensor

$$W_{\mu\nu} \equiv \frac{1}{8\pi}\sum_{\text{spins }\sigma}\sum_X < N(p,\sigma) \mid J_\mu(0) \mid X > < X \mid J_\nu(0) \mid N(p,\sigma) > \times (2\pi)^4\delta^4(p_X - q - p)\,.$$

- Symmetry property for hadronic tensor
  - Spin average $W_{\mu\nu}^{(\mathrm{em})} = W_{\nu\mu}^{(\mathrm{em})}$
  - Time-reversal invariance $W_{\mu\nu} = W_{\mu\nu}^{*}$
  - Current conservation $q^{\mu} W_{\mu\nu} = 0$
  - Two independent structure functions

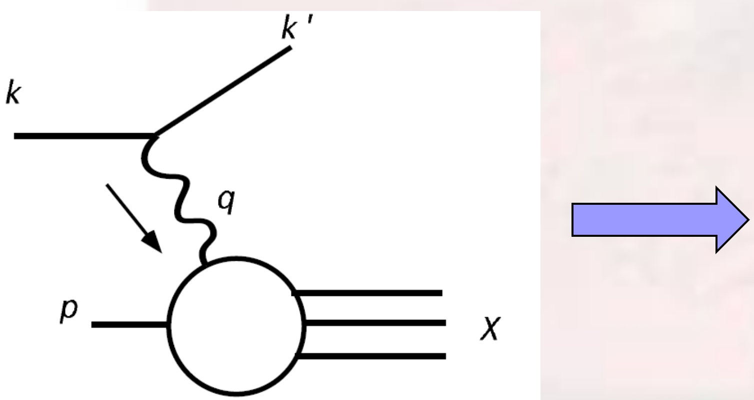
$$
\begin{aligned}
W_{\mu\nu}{}^{(em)} &= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(x, q^2) \\
&\quad + \left(p_\mu + q_\mu\left(\frac{1}{2x}\right)\right)\left(p_\nu + q_\nu\left(\frac{1}{2x}\right)\right) W_2(x, q^2)
\end{aligned}
$$

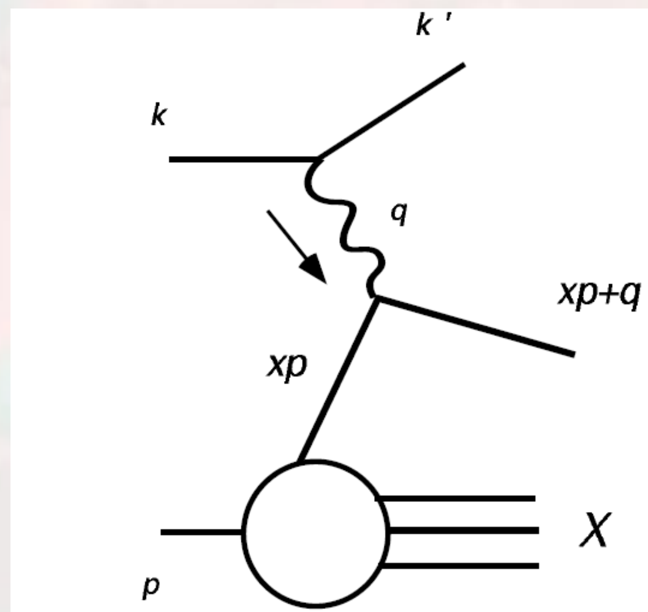
$$
= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F_2(x, Q^2) \qquad \hat{P}_\mu = P_\mu - \frac{P \cdot q}{q^2} q_\mu
$$

Naïve Parton Model

$$d\sigma^{(\ell N)}(p,q) = \sum_f \int_0^1 d\xi \; d\sigma_{\text{Born}}{}^{(\ell f)}(\xi p, q)\phi_{f/N}(\xi)$$

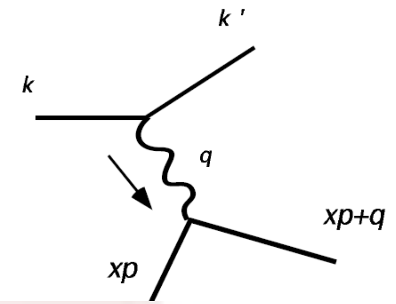
- $\phi_{f/N}(\xi)$ the parton distribution describes the probability that the quark carries nucleon momentum fraction





Naïve Parton Model

$$d\sigma^{(\ell N)}(p,q) = \sum_f \int_0^1 d\xi \; d\sigma_{\rm Born}{}^{(\ell f)}(\xi p, q)\phi_{f/N}(\xi)$$



- Partonic tensor is calculated

$$W_{\mu\nu}{}^{(f)} = \frac{1}{8\pi}\int \frac{d^3p'}{(2\pi)^3 2\omega_{p'}} Q_f{}^2 tr[\gamma_\mu \not{p}' \gamma_\nu \not{p}]\,(2\pi)^4\delta^4(p'-\xi p - q)$$

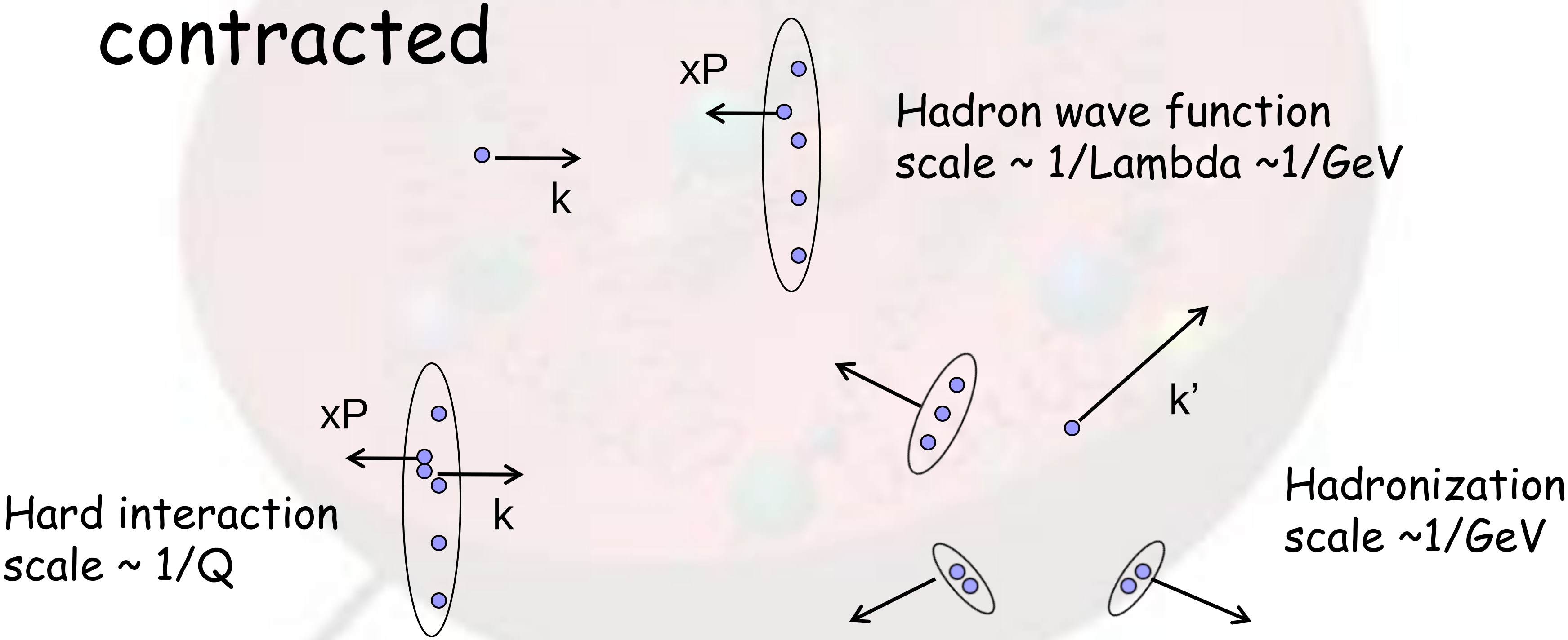
- Structure functions (home work)

$$F_2{}^{(N)}(x) = \sum_f Q_f{}^2 x\phi_{f/N}(x) = 2xF_1{}^{(N)}(x)$$

  - Callan-Gross relation: $F_2 = 2xF_1$
  - Quark spin is ½

Intuitive argument for the factorization (DIS)

- In the Bjorken limit, nucleon is Lorentz contracted



Factorization formula

$$F_2{}^{(h)}(x,Q^2) = \sum_{i=f,\bar{f},G} \int_x^1 d\xi\; C_2{}^{(i)}\left(\frac{x}{\xi},\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right)\phi_{i/h}(\xi,\mu^2)$$

$$F_1{}^{(h)}(x,Q^2) = \sum_{i=f,\bar{f},G} \int_x^1 \frac{d\xi}{\xi}\, C_1{}^{(i)}\left(\frac{x}{\xi},\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right)\phi_{i/h}(\xi,\mu^2)$$

- Factorization → scale dependence

$$\mu\frac{d^2}{d\mu^2}\phi_{i/h}(x,\mu^2) = \sum_{j=f,\bar{f},G}\int_x^1 \frac{d\xi}{\xi}P_{ij}(\frac{x}{\xi},\alpha_s(\mu^2))\phi_{j/h}(\xi,\mu^2)$$

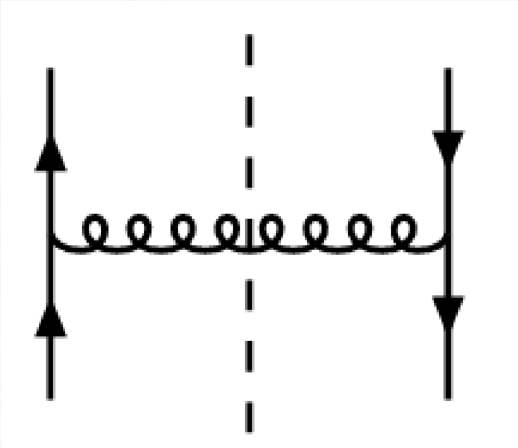
$$\mu\frac{d}{d\mu}\ln\bar{\phi}\left(n,\alpha_s(\mu^2)\right) = -\gamma_n\left(\alpha_s(\mu^2)\right) \qquad \bar{f}(n)\equiv\int_0^1 dx\; x^{n-1}f(x)$$

- Scale dependence → resummation

$$\bar{\phi}^{(\text{val})}(n,\mu^2) = \bar{\phi}^{(\text{val})}(n,\mu_0{}^2)\exp\left\{-\frac{1}{2}\int_0^{\ln\mu^2/\mu_0{}^2} dt\;\gamma_n\left(\alpha_s(\mu_0{}^2e^t)\right)\right\}$$

anomalous dimension: $\int_0^1 d\xi\,\xi^{n-1}P_{ij}(\xi,\alpha_s) = -\gamma_{ij}(n)$

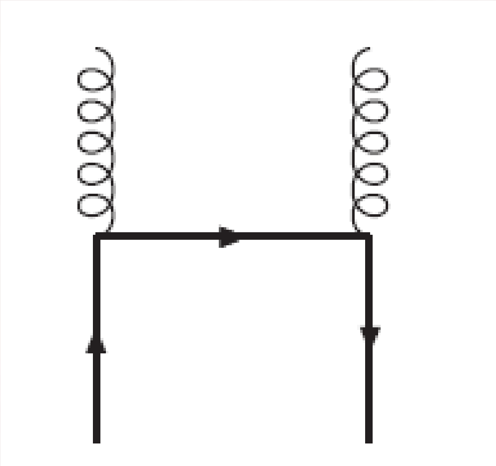
Quark-quark splitting



- Physical polarization for the radiation gluon
- Incoming quark on-shell, outgoing quark off-shell

$$\mathcal{P}_{qq} = C_F \left[\frac{1+x^2}{(1-x)_+} + \delta(1-x) \right]$$

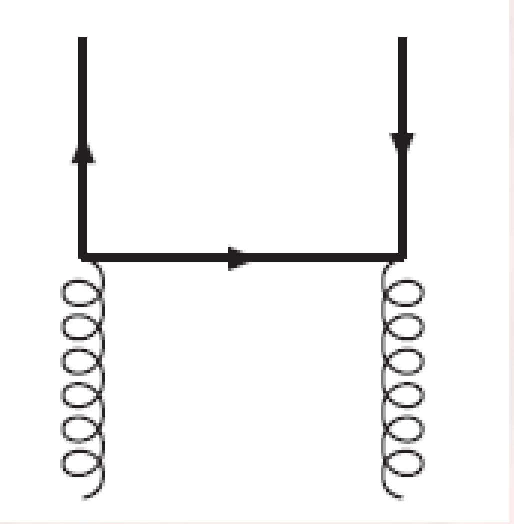
Quark-gluon splitting



- Incoming quark on-shell, gluon is off-shell

$$\mathcal{P}_{g/q} = C_F \left[\frac{1+(1-x)^2}{x} \right]$$

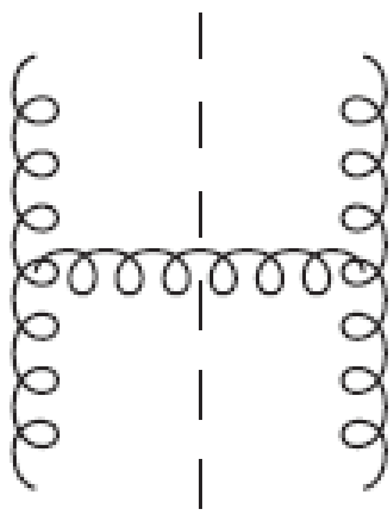
Gluon-quark splitting



- Incoming gluon is on-shell, physical polarization

$$\mathcal{P}_{q/g} = T_F\left[(1-x)^2 + x^2\right]$$

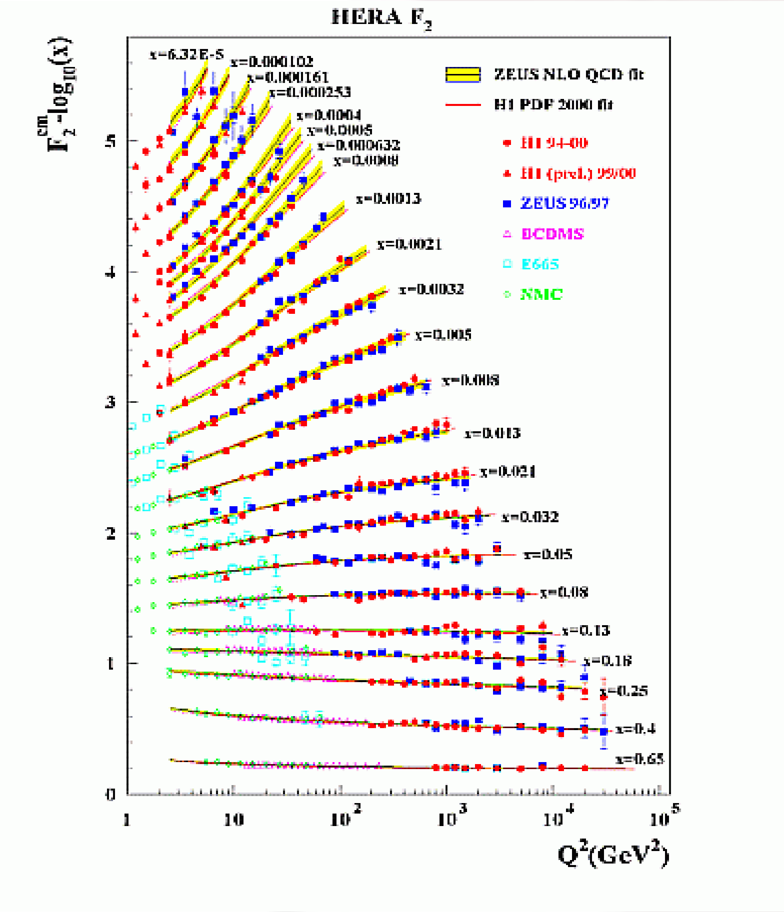
Gluon-gluon splitting

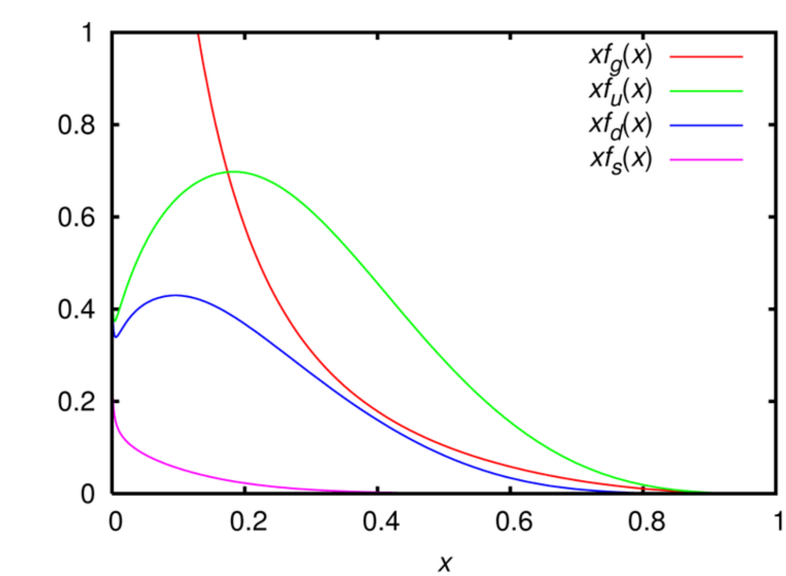


- Physical polarizations for the gluons

$$\mathcal{P}_{gg}(x) = \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) + \delta(x-1)\beta_0$$

These evolutions describe the HERA data





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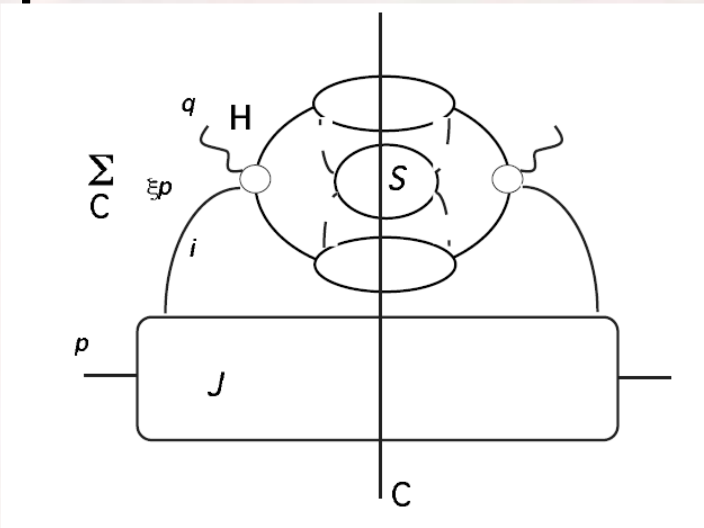
- Let's follow the example of e+e- annihilation, to discuss the factorization for DIS
- First the pinch singular surfaces

Pinch singular surfaces for DIS

- Physical picture

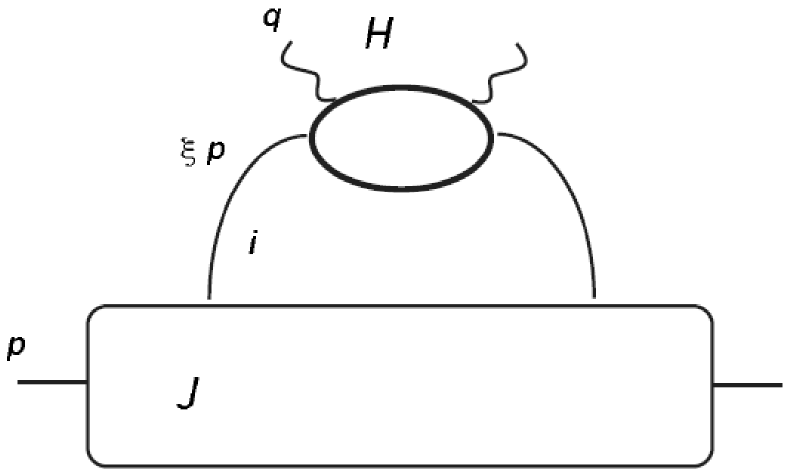


  - Sum over all final states, soft/collinear divergences associated with final state jets cancel

- Hadronic tensor can be written as imaginary part of the forward scattering amplitude

$$W_{\mu\nu} = 2\,\mathrm{Im}\,T_{\mu\nu}$$

$$T_{\mu\nu} = \frac{i}{8\pi}\int d^4x\, e^{iq\cdot x} < h(p)|\, T\, J_\mu(x)J_y(0)\, |h(p) >$$



Factorization

- Parton model form

$$W^{\mu\nu} = \sum_i \int_x^1 d\xi\, H^{\mu\nu}_{i,\beta\alpha}(q,\xi p) J_{i,\alpha\beta}(\xi)$$

  - Long distance function, connecting the hadron to the hard part

$$J_{q,\alpha\beta} = \frac{1}{2}\sum_{\substack{\text{spin}\\ \sigma}} \int_{-\infty}^{\infty} \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} < h(p,\sigma) \mid \bar{q}_\alpha(0^+, y^-, \mathbf{0}_\perp) q_\beta(0) \mid h(p,\sigma) >$$

    - Product of operators, separated by the light-like distance

- Leading order projection is the un-polarized parton distribution

$$\phi_{q/h}(\xi,\mu^2) = \frac{1}{2}\sum_\sigma \int_{-\infty}^{\infty} \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} < h(p,\sigma) \mid \bar{q}(0^+, y^-, \mathbf{0}_\perp)\, \frac{1}{2} n\cdot\gamma\, q(0) \mid h(p,\sigma) >,$$

$$\phi_{G/h}(\xi,\mu^2) = \frac{1}{4\pi\xi p^+}\int dy^- e^{-i\xi p^+ y^-} \sum_\sigma \sum_{\mu=1}^{2} < h(p,\sigma) \mid F^{+}{}_{\mu}(0, y^-, \mathbf{0}) F^{\mu +}(0) \mid h(p,\sigma) >$$

- They are expressions in the physical gauge → gauge invariant definition